UNIQUENESS THEOREMS FOR THE SOLUTION OF THE INVERSE PROBLEM OF HEAT CONDUCTION

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UDC 532.526.011:536.24.02

Uniqueness conditions are found for the solution of the problem of reproducing the thermophysical properties of a nonlinear heat-conducting medium by means of a known nonstationary temperature field. The results can be used in practical methods of determining the thermophysical properties of materials.

1°. FORMULATION OF THE PROBLEM

Let us consider a one-dimensional nonstationary heat-conduction problem:

$$c\rho \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right); \; t_0 \leqslant t \leqslant t_1, \; x_0 < x < x_1,$$

$$T(x, t_0) = T^0(x), \; T(x_0, t) = T_0(t), \; T(x_1, t) = T_1(t).$$

The coefficients $c\rho$ and λ are assumed positive and, respectively, continuous and continuously differentiable functions of T. Making the variables dimensionless, let us write the problem as

$$\frac{\partial u}{\partial \tau} - \psi \frac{\partial}{\partial x}\left(\varphi \frac{\partial u}{\partial x}\right) = 0; \; (x, \tau) \in \bar{I}^2, \; I \equiv (0, 1),$$

$$u(0, \tau) = f_0(\tau), \; u(1, \tau) = f_1(\tau), \; u(x, 0) = f(x). \tag{1}$$

The boundary and initial functions f_0, f_1, f are assumed to satisfy the first-order compatibility conditions [1]: there exists a function $v(x, \tau)$ ($v \in C^{2,1}$) twice continuously differentiable with respect to x and continuously differentiable with respect to τ such that

$$v(0, \tau) = f_0(\tau), \; v(1, \tau) = f_1(\tau), \; v(x, 0) = f(x),$$

$$\lim_{\tau \to 0+}\left\{\frac{\partial v}{\partial \tau} - \psi(v) \frac{\partial}{\partial x}\left(\varphi(v) \frac{\partial v}{\partial x}\right)\right\} = 0, \; x \in \bar{I}. \tag{2}$$

The conditions listed assure the existence and uniqueness of the solution. $u \in C^{2,1}$ of the problem (1) for any pair of functions (φ, ψ), i.e., a mapping P can be introduced from the set M in the class $C^{2,1}$:

$$P(\varphi, \psi) = u(x, \tau); \; M \equiv \{(\varphi, \psi) | (\varphi, \psi) \in C^0 \times C^1; \; \varphi > 0, \; \psi > 0\}. \tag{3}$$

We shall understand the solution of the inverse problem to be any solution of the operator equation (3) in the pair (φ, ψ) for a given solution $u \in C^{2,1}$:of the problem (1) and the conditions (2). Let us note at once the evident ambiguity of the solution (φ, ψ)

$$P(\varphi, \psi) = P\left(k\varphi, \; \frac{1}{k}\psi\right), \; k > 0.$$

It is impossible to eliminate an ambiguity of this kind by some selection of the solution u; two solutions of (3) connected by the relationship

$$\tilde{\psi} = k\psi, \; \varphi\psi = \tilde{\varphi}\tilde{\psi}, \tag{4}$$

N. E. Zhukovskii Central Aero-Hydrodynamics Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 1, pp. 145-150, July, 1975. Original article submitted January 17, 1975.

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will be called equivalent. Let the function f(x) governing the initial distribution u(x, 0) be varied, wherein the compatibility conditions (2) will not be spoiled ($f \in N$). Keeping in mind the variation of f, let us introduce the notation P_f for the mapping

$$P_f(\varphi,\ \psi):\ (f,\ \varphi,\ \psi)\to u(x,\ \tau),\ f\in N.$$

2°. UNIQUENESS CONDITIONS

The following theorem expressing the independence of the thermophysical characteristics $(\varphi,\ \psi)$ from the initial distribution will hold.

THEOREM 1. If $(\varphi,\ \psi)$, $(\tilde{\varphi},\ \tilde{\psi})$ are two solutions of (3) such that for any function f differentiable a sufficient number of times, $P_f(\varphi,\ \psi)$ agrees with $P_f(\tilde{\varphi},\ \tilde{\psi})$, then $(\varphi,\ \psi)$ and $(\tilde{\varphi},\ \tilde{\psi})$ are equivalent:

$$\tilde{\psi}=k\psi,\ \varphi=k\tilde{\varphi}.$$

Proof. For $\tau\to 0^+$ we have from (1)

$$\frac{d}{dx}\left(\varphi\frac{df}{dx}\right)=\zeta\frac{d}{dx}\left(\tilde{\varphi}\frac{df}{dx}\right),\ \zeta\equiv\tilde{\psi}/\psi \tag{5}$$

or

$$\frac{d}{dx}\left(\varphi\frac{df}{dx}-\zeta\tilde{\varphi}\frac{df}{dx}\right)+\frac{d\zeta}{df}\tilde{\varphi}\left(\frac{df}{dx}\right)^2=0. \tag{6}$$

The variation of (6) yields $[\mu\equiv\tilde{\varphi}(d\zeta/df)]$

$$2\mu\frac{df}{dx}\cdot\frac{d\delta f}{dx}+\frac{d\mu}{df}\left(\frac{df}{dx}\right)^2\delta f+\frac{d}{dx}\left(\varphi\frac{d\delta f}{dx}-\zeta\tilde{\varphi}\frac{d\delta f}{dx}\right)=0. \tag{7}$$

The relationship (7) is satisfied identically for sufficiently smooth variations δf. Integrating (7) with respect to x between 0 and 1 and using the known Dubois–Raymond lemma, we obtain

$$\frac{d\mu}{df}\left(\frac{df}{dx}\right)^2+2\mu\frac{d^2f}{dx^2}=0,\ f\in N. \tag{8}$$

A variation of the initial equality (5) yields

$$\frac{d}{dx}\left(\tilde{\varphi}\frac{d\delta f}{dx}+\frac{d\varphi}{df}\cdot\frac{df}{dx}\delta f\right)=\zeta\frac{d}{dx}\left(\tilde{\varphi}\frac{d\delta f}{dx}+\frac{d\tilde{\varphi}}{df}\cdot\frac{df}{dx}\delta f\right)+\frac{d\zeta}{df}\cdot\frac{d}{dx}\left(\tilde{\varphi}\frac{df}{dx}\right)\delta f.$$

Hence, we have for $\delta f\equiv a$, in particular,

$$\frac{d}{dx}\left(\frac{d\varphi}{df}\cdot\frac{df}{dx}\right)=\zeta\frac{d}{dx}\left(\frac{d\tilde{\varphi}}{df}\cdot\frac{df}{dx}\right)+\frac{d\zeta}{df}\cdot\frac{d}{dx}\left(\tilde{\varphi}\frac{df}{dx}\right),$$

but

$$\zeta\frac{d}{dx}\left(\frac{d\tilde{\varphi}}{dx}\right)+\frac{d\zeta}{df}\left(\frac{d\tilde{\varphi}}{dx}\cdot\frac{df}{dx}+\tilde{\varphi}\frac{d^2f}{dx^2}\right)=\frac{d}{dx}\left(\zeta\frac{d\tilde{\varphi}}{dx}\right)+\mu\frac{d^2f}{dx^2}.$$

Therefore, we obtain the equality

$$\frac{d}{dx}\left[\left(\frac{d\varphi}{df}-\zeta\frac{d\tilde{\varphi}}{df}\right)\frac{df}{dx}\right]=\mu\frac{d^2f}{dx^2}. \tag{9}$$

Variating (9) and again using the Dubois–Raymond lemma, we arrive at the new relationship

$$\frac{d\mu}{df}\cdot\frac{d^2f}{dx^2}+\frac{d^2\mu}{dx^2}=0$$

or

$$2\frac{d\mu}{df}\cdot\frac{d^2f}{dx^2}+\frac{d^2\mu}{df^2}\left(\frac{df}{dx}\right)^2=0. \tag{10}$$

Differentiating (9) with respect to x permits obtaining still another equality

$$\frac{df}{dx}\left[\frac{d^2\mu}{df^2}\left(\frac{df}{dx}\right)^2+2\frac{d\mu}{df}\cdot\frac{d^2f}{dx^2}\right]+2\frac{d}{dx}\left(\mu\frac{d^2f}{dx^2}\right)=0,$$

which denotes together with (10) that

$$\frac{d}{dx}\left(\mu\frac{d^2f}{dx^2}\right)=0. \tag{11}$$

If (11) is varied and we set $\delta f = a$ and $\delta f = bx$ successively, then we can obtain the identity

$$\frac{d\mu}{df}\equiv 0,$$

i.e.,

$$\mu=\text{const}. \tag{12}$$

The relationships (12) and (8) result in the equality $\mu = 0$. The first part of the equivalence condition $\tilde{\psi} = k\psi$ has therefore been proved. Now (5) can be written as

$$\frac{d}{dx}\left(\sigma\frac{df}{dx}\right)=0,\ \text{where}\ \sigma=\varphi-k\tilde{\varphi}\equiv\sigma(f). \tag{13}$$

If (13) is varied and we set $\delta f = a$ and $\delta f = bx$ successively, then we can obtain the equality

$$\frac{d}{dx}\left(\frac{d\sigma}{df}\cdot\frac{df}{dx}\right)=0,\quad \frac{d}{dx}\left(\sigma+x\frac{d\sigma}{df}\cdot\frac{df}{dx}\right)=0$$

or

$$2\frac{d\sigma}{df}\cdot\frac{df}{dx}+x\frac{d}{dx}\left(\frac{d\sigma}{df}\cdot\frac{df}{dx}\right)=0,$$

where the relationship

$$\frac{d\sigma}{dx}=0,\ f\in N,\ \text{i.e.,}\ \sigma=\text{const}.$$

is the consequence of the last three equations. Finally, turning again to (13), we obtain

$$\sigma\equiv 0=\varphi-k\tilde{\varphi}.$$

The equivalence of $(\varphi,\ \psi)$ and $(\tilde{\varphi},\ \tilde{\psi})$ has thereby been proved.

Let us note that although the conditions of Theorem 1 can be shown to be quite rigorous mathematically, they are physically natural, since if $(\varphi,\ \psi)$ have the meaning of physical characteristics of a medium, then they should be invariant relative to the selection of the initial temperature distribution. Theorem 1 establishes the possibility of a unique (to the accuracy of an equivalency) reproduction of the thermophysical properties of a medium in the presence of a sufficiently complete set of external effects, since every instantaneous distribution which is realized as a reaction to some external effects f_0, f_1 can be taken as the initial one for all the subsequent times. The necessary and sufficient condition for uniqueness of the solution of (3) as a property of the temperature field $u(x,\ \tau)$ is presented below. There holds the following theorem:

THEOREM 2. The solution $(\varphi,\ \psi)$ of the problem (3) is ambiguous if and only if the field $u(x,\ \tau)$ has the form $u = u(z)$. The function $z(x,\ \tau)$ is determined by one of the equalities

$$z=ax+b\tau+c\quad\text{or}\quad z=\frac{x+b}{\sqrt{a(c-\tau)}}+d. \tag{14}$$

Here a, b, c, d are arbitrary constants.

Necessity. Let there be two solutions

$$\frac{\partial u}{\partial\tau}-\psi_1\frac{\partial}{\partial x}\left(\varphi_1\frac{\partial u}{\partial x}\right)=\frac{\partial u}{\partial\tau}-\psi_2\frac{\partial}{\partial x}\left(\varphi_2\frac{\partial u}{\partial x}\right)=0. \tag{15}$$

It follows from (15) that

$$(\psi_1\varphi_1-\psi_2\varphi_2)\frac{\partial^2u}{\partial x^2}=(\psi_2\varphi_2'-\psi_1\varphi_1')\left(\frac{\partial u}{\partial x}\right)^2, \tag{16}$$

where neither $(\psi_1\varphi_1 - \psi_2\varphi_2)$ nor $(\psi_2\varphi_2' - \psi_1\varphi_1')$ can vanish identically, since it would hence follow that $(\varphi_1,\ \psi_1)$ and $(\varphi_2,\ \psi_2)$ are equivalent.

Equation (16) is converted into

$$\frac{\partial}{\partial x}\left(1\Big/\frac{\partial u}{\partial x}\right)=F_1(u). \tag{17}$$

If a new function z(u) is introduced by the relationship

$$\frac{dz}{du}=\exp\left(\int_0^n F_1(v)\,dv\right),$$

then the equation

$$\frac{\partial}{\partial x}\left(1\Big/\frac{\partial z}{\partial x}\right)=0$$

is easily obtained from (17). We therefore find that $u(x, \tau)$ should be

$$u=u[\alpha(\tau)x+\beta(\tau)]. \tag{18}$$

To determine the functions $\alpha(\tau)$ and $\beta(\tau)$ let us substitute (18) into (15):

$$\dot{\alpha}x+\dot{\beta}=\alpha^2\cdot F_2(u);\ F_2=\psi_1(u)\frac{du}{dz}(\varphi_1'-\varphi_1F_1).$$

It is easy to see that F_2 should depend linearly on z. We finally have the system of equations

$$\dot{\alpha}-\xi\alpha^3=0,\ \dot{\beta}-\alpha^2(a\beta-\eta)=0. \tag{19}$$

Here ξ and η are arbitrary constants. The expressions (14) for $z(x, \tau)$ are obtained by integrating (19) in conformity with the cases $\xi = 0$, $\xi \neq 0$. The necessity is proved.

Sufficiency. Let $u = u(z)$. It is established by direct substitution that (1) reduces to

$$\frac{d}{dz}\left(\varphi\frac{du}{dz}\right)=\frac{b}{a\psi}\cdot\frac{du}{dz},$$

if $z(x, \tau)$ corresponds to the case $\xi = 0$, or to

$$\frac{d}{dz}\left(\varphi\frac{du}{dz}\right)=\frac{a(z-d)}{2\psi}\cdot\frac{du}{dz}$$

otherwise. In both cases the ambiguity of the solution of the problem (3) is evident. Theorem 2 is completely proved.

Using the equality of the corresponding Jacobians to zero, let us introduce quantitative criteria assuring the uniqueness of the solution of the problem (3):

$$i_1=\inf_{\alpha,\beta}\left\{\max_{x,\tau}\left|\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cdot\frac{\partial u}{\partial\tau}-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\cdot\frac{\partial u}{\partial x}\right|\right\},\ (x,\tau)\in\bar{I}^2,$$

$$i_2=\inf_{\alpha,\beta}\left\{\max_{x,\tau}\left|\frac{x+\alpha}{\sqrt{\alpha^2+\beta^2}}\cdot\frac{\partial u}{\partial x}+2\frac{\tau+\beta}{\sqrt{\alpha^2+\beta^2}}\cdot\frac{\partial u}{\partial\tau}\right|\right\},\ \alpha^2+\beta^2>0.$$

By using the criteria i_1, i_2, Theorem 2 can be formulated as follows: the problem (3) has a unique solution if and only if $J \equiv i_1 \cdot i_2 > 0$.

The assertion of Theorem 2 can be clarified qualitatively as follows. If the temperature field $u(x, \tau)$ reflects some symmetry property inwardly inherent to (1), then the solution of problem (3) turns out to be ambiguous. It is known that the symmetry properties of an equation are described in the most general manner by means of transformation group theory. It turns out that in the case of the heat-conduction equation the class of group-invariant solutions [2] corresponding to all single-parameter subgroups of transformations admitted by (1) agrees exactly with the class of temperature fields yielding an ambiguous solution of the problem (3). As has been established by Ovsyannikov [2], Eq. (1) for the arbitrary dependences $\varphi(u)$, $\psi(u)$ admits of a fundamental group of transformations with a Lie algebra of infinitesimal operators with the basis

$$X_1=\frac{\partial}{\partial\tau},\ X_2=\frac{\partial}{\partial x},\ X_3=2\tau\frac{\partial}{\partial\tau}+x\frac{\partial}{\partial x}-q\frac{\partial}{\partial q},$$

here $q \equiv \varphi(\partial u/\partial x)$ is the heat flux. Let us examine the following family of elements:

$$X^*(\beta, \ \varepsilon) = 2(\tau + \beta)\frac{\partial}{\partial \tau} + (x + \varepsilon)\frac{\partial}{\partial x} - q\frac{\partial}{\partial q},$$

in operator algebra, where β, ε are arbitrary constants. The following functions:

$$J_1 \equiv z = \frac{ax + b}{\sqrt{\tau + c}} + d, \ J_2 = u, \ J_3 = q(x + b/a)$$

can evidently be selected as the complete system of invariants of the operator X^*. Omitting solutions of the form $u(x)$ or $u(\tau)$ as not being of interest, let us also consider the family

$$X^{**}(\beta, \ \varepsilon) = \beta X_1 + \varepsilon X_2.$$

The functions

$$J_1 \equiv z = ax + b\tau + c, \ J_2 = u, \ J_3 = q$$

can be taken as the invariants X^{**}. Invariant solutions corresponding to the arbitrary operator X are sought for in the form of dependences between the invariants of this operator [2].

Therefore, we obtain a new formulation of the necessary and sufficient condition for uniqueness of the problem (3) in terms of the group properties of the heat conduction equation.

THEOREM 3. The inverse problem of heat conduction has an ambiguous solution if and only if the temperature field $u(x, \tau)$ is an invariant solution corresponding to the one-parameter subgroup of the fundamental group admitted by the heat-conduction equation for the arbitrary dependences $\varphi(u)$, $\psi(u)$.

LITERATURE CITED

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